DOCUMENT RESUME

ED 441 662 SE 063 387

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TITLE	Teaching and Learning a New Algebra with Understanding.
INSTITUTION	National Center for Improving Student Learning and Achievement in Mathematics and Science, Dartmouth, MA.
SPONS AGENCY	Office of Educational Research and Improvement (ED), Washington, DC.; National Science Foundation, Arlington, VA.
PUB DATE	2000-00-00
NOTE	34p.
CONTRACT	R305A60007-98; RED9619102
PUB TYPE	Opinion Papers (120)
EDRS PRICE	MF01/PC02 Plus Postage.
DESCRIPTORS	*Algebra; *Educational Change; Elementary Education; Learning Strategies; Mathematics Instruction; Teaching Methods

ABSTRACT

This paper suggests a route to deep, long-term algebra reform that begins not with more new approaches but with elementary school teachers and the reform efforts that currently exist. This route involves generalization and expression of that generality using increasingly formal languages, beginning with arithmetic, modeling situations, geometry, and virtually all mathematics that can or should appear in the elementary grades. This route involves infusing algebra throughout the mathematics curriculum from the very beginning. A model of this approach, suggestions, and examples are provided. (Contains 10 references.) (CCM)

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NATIONAL CENTER FOR IMPROVING STUDENT LEARNING
& ACHIEVEMENT IN MATHEMATICS & SCIENCE

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Teaching and Learning a New Algebra With Understanding[1]

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The research reported herein was supported in part by a grant from the U.S. Department of Education, Office of Educational Research and Improvement, to the National Center for Improving Student Learning and Achievement in Mathematics and Science (R305A60007-98). The opinions expressed herein do not necessarily reflect the position, policy, or endorsement of the supporting agencies.

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To discuss the teaching and learning of algebra with understanding, we must first look at the algebra students too often encounter in their classrooms. The traditional image of algebra, based in more than a century of school algebra, is one of simplifying algebraic expressions, solving equations, learning the rules for manipulating symbols—the algebra that almost everyone, it seems, loves to hate. The algebra behind this image fails in virtually all the dimensions of understanding that Carpenter and Lehrer (this volume) have taken as a starting point for reform in the classroom. School algebra has traditionally been taught and learned as a set of procedures disconnected both from other mathematical knowledge and from students' real worlds.

Construction of relationships and application of newly acquired knowledge are not at the heart of traditional algebra: The "applications" used are notoriously artificial (e.g., "age problems" and "coin problems"), and students are neither given the opportunity to reflect on their experiences nor the support to articulate their knowledge to others. Instead, they memorize procedures that they know only as operations on strings of symbols, solve artificial problems that bear no meaning to their lives, and are graded not on understanding of the mathematical concepts and reasoning involved, but on their ability to produce the right symbol string—answers about which they have no reason to reflect and that they found (or as likely guessed) using strategies they have no need to articulate. Worst of all, their experiences in algebra too often drive them away from mathematics before they have experienced not only their own ability to construct mathematical knowledge and to make it their own, but, more importantly, to understand its importance–and usefulness–to their own lives.

Although algebra has historically served as a gateway to higher mathematics, the gateway has been closed for many students in the United States, who are shunted into academic and career dead ends as a result. And even for those students who manage to pass through the gateway, algebra has been experienced as an unpleasant, even alienating event - mostly about manipulating symbols that don't stand for anything. On the other hand, algebraic reasoning in its many forms, and the use of algebraic representations such as graphs, tables, spreadsheets and

traditional formulas, are among the most powerful intellectual tools that our civilization has developed. Without some form of symbolic algebra, there could be no higher mathematics and no quantitative science, hence no technology and modern life as we know them. Our challenge then is to find ways to make the power of algebra (indeed, all mathematics) available to all students—to find ways of teaching that create classroom environments that allow students to learn with understanding. The broad outlines of the needed changes follow from what we already know about algebra teaching and learning:

- begin early (in part, by building on students' informal knowledge),
- integrate the learning of algebra with the learning of other subject matter (by extending and applying mathematical knowledge),
- include the several different forms of algebraic thinking (by applying mathematical knowledge),
- build on students' naturally occurring linguistic and cognitive powers (encouraging them at the same time to reflect on what they learn and to articulate what they know), and
- encourage active learning (and the construction of relationships) that puts a premium on sense-making and understanding.

Making these changes, however, will not be easy, especially where the new approaches involve new tools, unprecedented applications, populations of students traditionally not targeted to learn algebra, and K-8 teachers traditionally not educated to teach algebra (neither the old algebra nor some new version). Despite these challenges, this chapter suggests a route to deep, long-term algebra reform that begins not with more new-fangled approaches but with the elementary school teachers and the reform efforts that currently exist. This route involves generalizing and expressing that generality using increasingly formal languages, where the generalizing begins in arithmetic, in modeling situations, in geometry, and in virtually all the

mathematics that can or should appear in the elementary grades. Put bluntly, this route involves infusing algebra throughout the mathematics curriculum from the very beginning of school.

Although this chapter is designed to show this route to teaching for understanding in greater detail, I have chosen to organize the material around the five different forms of algebraic reasoning as I see them (see Figure 1) in order to demonstrate how algebra can infuse and enrich most mathematical activity from the early grades onward. These five interrelated forms, each discussed in the sections that follow, form a complex composite. The first two of these underlie all the others, the next two constitute topic strands, and the last reflects algebra as a web of languages and permeates all the others. All five richly interact conceptually as well as in activity —To understand this algebra is to make a rich web of connections. The classroom examples in these sections are based in actual student work and language and are taken from across many grade levels and mathematical topic areas. Together, the forms of reasoning and the classroom examples discussed in this chapter emphasize where we need to go rather than where we are, or have been.

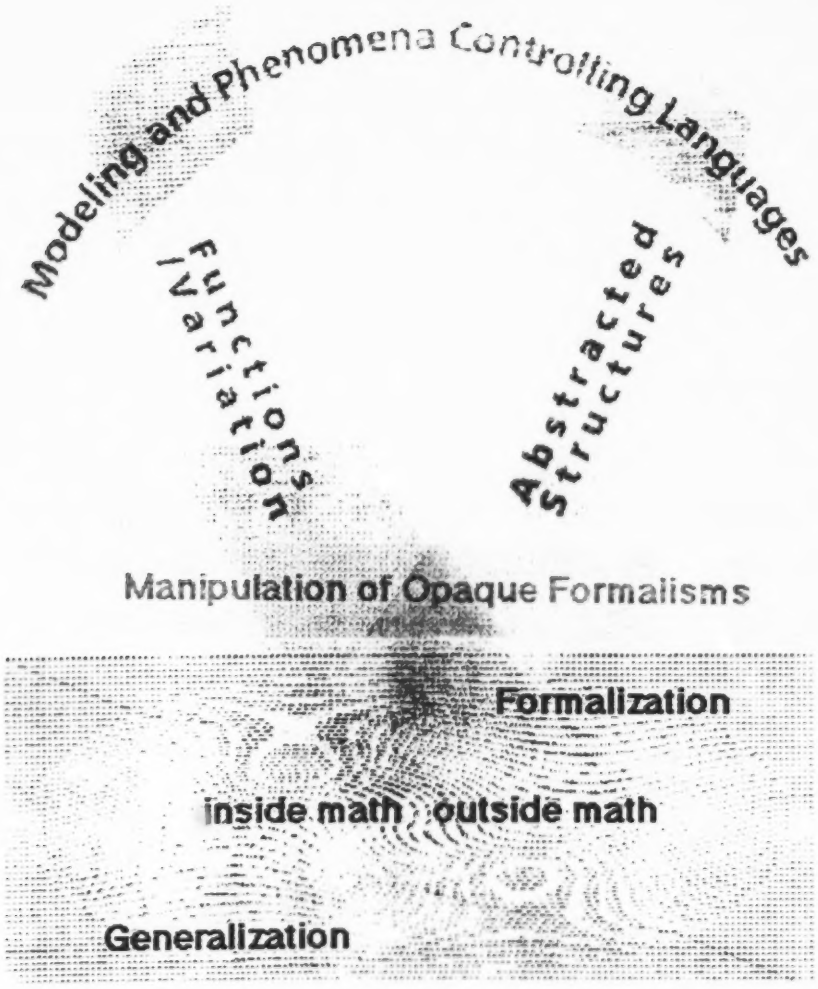

Figure 1. The overlapping and interrelationships of the five forms of algebraic reasoning.

Algebra as the Generalization and Formalization of Patterns and Contraints

Although pure computational arithmetic of the sort that dominates elementary school mathematics, the kinds of counting and sorting involved in combinatorics, and pure spatial visualization need not inherently emphasize generalization and formalization, it is difficult to point to mathematical systems and situational contexts where mathematical activity does not

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involve these two processes. The manipulations performed on formalisms (which I identify in this chapter as the second kernel aspect of algebra and which sometimes yield general patterns and structures—the essence of the third, structural, aspect of algebra) typically occur as the direct or indirect result of prior formalization. Generalization and formalization are intrinsic to mathematical activity and thinking—they are what make it mathematical.

Generalization involves deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases, or lifting the reasoning or communication to a level where the focus is no longer on the cases or situations themselves, but rather on the patterns, procedures, structures, and the relations across and among them (which, in turn, become new, higher level objects of reasoning or communication). But expressing generalizations means rendering them into some language, whether in a formal language, or, for young children, in intonation and gesture. In the case of young children, identifying the expressed generality or the child's intent that a statement about a particular case be taken as general may require the skilled and attentive ear of a teacher who knows how to listen carefully to children.

We distinguish two sources of generalization and formalization: reasoning and communication in mathematics proper, usually beginning in arithmetic; and reasoning and communicating in situations based outside mathematics but subject to mathematization, usually beginning in quantitative reasoning. The distinction between these two sources (mathematics proper and situations outside mathematics) is especially problematic in the early years, when mathematical activity takes very concrete forms and is often tightly linked to situations that give rise to the mathematical activity. Whether the starting point is in mathematics (and therefore arising from previously mathematized experience) or from a yet-to-be-mathematized situation, the source is ultimately based in phenomena or situations outside mathematics proper because, after all, mathematics thinking ultimately arises from experience and only becomes mathematical upon appropriate activity and processing. This view is the basis of many reform curricula.

Although students in traditional mathematics classrooms might model the same situations as students in classrooms that promote understanding, and reason similarly within mathematics to formalize those situations into algebraic differences and inequalities, students in traditional classrooms are more likely to generalize from objects and relations already conceived as mathematical (e.g., a student might generalize patterns in sequences of numbers in a hundreds table or multiplication table—mathematics proper). In classrooms that promote understanding, students are more likely to begin by generalizing from their conceptions of situations experienced as meaningful and to derive their formalizations from conceptual activities based in those situations. For example, a student comparing differences in prices between cashews (expensive) and peanuts (cheap) for two different brands, might generalize that a small increase in price of, say, peanuts of brand A will not change the outcome of a comparison.This difference of context can make mathematics involved more vital/important to the student both in the short and long term.

A Classroom Example of Early Generalization and Formalization

The following example from a third-grade class was observed and documented by Virginia Bastable and Deborah Schifter (in press). The teacher began by asking how many pencils there were in three cases, each containing 12 pencils. After the class arrived at a repeated-addition (12 + 12 + 12) solution, the teacher showed how the result could be seen as a (3 _ 12) multiplication. She expected to move on to a series of problems of this type, but one student noted that each 12 could be decomposed into two 6s, and that the answer could be described as 6 + 6 + 6 + 6 + 6 + 6 or six 6s and could be written as 6×6. Another student observed that each 6 could also be thought of as two 3s, yielding twelve 3s or 12×3. Another student realized that "This one is the backwards of our first one, 3×12." What follows is a description of the extended investigation that occurred.

Activity. The students continued to find ways of grouping numbers that totaled 36. One student looking at the column of 3s, suggested four groups of three 3s, or 4×9. Another student noted that "we can add another one to the list because if $4 \times 9 = 36$, then $9 \times 4 = 36$, too." One student objected, asking a question the teacher found interesting: "Does that always work? I mean, saying each one backwards will you always get the same answer?" When the teacher asked her what she thought, the student said, "I'm not sure. It seems to, but I can't tell if it would always work. I mean for all numbers."

For homework, the teacher asked them to explore ways to prove (or disprove) the student's question. The next day, the students explained their thinking, noting various number pairs such as 3×4 and 4×3 and sometimes using manipulatives to illustrate their examples. Although the original objector was still not convinced that this would work for all numbers, the teacher decided to leave the issue unresolved temporarily and continue exploration of multiplication by introducing arrays. Two weeks later, however, the teacher reintroduced the problem, suggesting students use what they now knew of arrays "to prove that the answer to a multiplication equation would be the same no matter which way it was stated." The class worked on this for a while, alone and with partners. Finally, one student decided she could prove it. Holding up three sticks of 7 Unifix cubes, she said,

> See, in this array I have three 7s. Now watch. I take this array [picking up the three 7-sticks] and put it on top of this array [turns them 90 degrees and places them on the seven 3-sticks she has previously arranged]. And look— they fit exactly. So 3×7 equals 7×3, and there's 21 in both. No matter which equation you do it for, it will always fit exactly.

At the end of this explanation, another student eagerly explained another way to prove it:

> I'll use the same equation as Lauren, but I'll only need one of the sets of sticks. I'll use this one [picks up the three 7-sticks]. When you look at it this way [holding the sticks up vertically], you have three 7s. But this way [turning the sticks sideways], you have seven 3s. See? . . . So this one array shows both 7×3 or 3×7.

At this explanation, the class objector agreed: Although both students had used a 3×7 array to explain their points, the final, simpler representation convinced her of the general claim. As she noted: "That's a really good way to show it . . . It would have to work for all numbers."

Discussion. In this example, students were attempting to generalize what they saw in a few cases of multiplication to all cases of multiplication and (because they had not yet worked with formal language in mathematics) to articulate their generalization through a variety of notational devices in combination with "natural," informal language. The basic issue was the range of the generalization—Did it hold for all numbers? The students used cubes and sticks to generate their ideas, to show one another their thinking, and to justify claims that were clearly theirs not their teacher's. The questions of certainty and justification arose as an integral aspect of the process and were interwoven in their use of notations. Thinking of this activity merely as the children developing the concept of commutativity of multiplication (of natural numbers) trivializes what happened during this extended lesson. The students were actually constructing both the very idea of multiplication (although only two aspects: repeated addition and array models) while beginning to develop the notion of mathematical justification and proof.

Although the episode began in a concrete situation, it quickly became a mathematical exploration. Pencils and cases were the stepping-off point that (inadvertently) led the students to the grouping and decomposition of whole numbers and, after some reflection, to the articulation of their newly constructed knowledge (the equivalence of alternative groupings) through use of concrete arrays of cubes. Students found ways to articulate the invariance of the "amount," or total, first under alternate groupings of 21 and then under alternate orientations of the same

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physical grouping. In the end, despite the fact that they did not have a formal language available, the generality was not only realized, but made explicit: "It would have to work for all numbers." It is easy to imagine that this property might be given a more formal expression later, first perhaps as "box times circle = circle times box," and then later as $\underline{a} \times \underline{b} = \underline{b} \times \underline{a}$.

Summary. The Bastable/Schifter study (in press) study from which the above case was taken includes several examples of such episodes (across Grades 1–6) involving properties of numbers (odd-even, zero), operations, extensions to other number systems beyond the natural numbers, and so on. Many important questions remain unanswered about these activities and how to organize them, including what the role(s) of language and special notations might be, how to discern generality in students' informal utterances, what the interplay between generalization and justification might be, what the role of concrete situations might be, and so on.

One other aspect of this situation deserves attention: This is certainly not traditional symbol-manipulation algebra. Although this was clearly an excellent teacher doing a good job in an arithmetic, this extended lesson focusing on generalization rather than computation took place in what many teachers would regard as the normal course of mathematical concept development in an "ordinary" mathematics classroom ("ordinary" in the sense of fitting the NCTM Professional Standards for Teaching Mathematics [1991]).

Algebra as Syntactically Guided Manipulation of (Opaque) Formalisms

When we deal with formalisms, whether traditional algebraic ones or those more exotic, our attention is on the symbols and syntactical rules for manipulating those formalisms rather than on what they might stand for, with much of their power arising from internally consistent, referent-free operations. The user suspends attention to what the symbols stand for and looks at the symbols themselves, thus freed to operate on relationships far more complex than could be managed if he or she needed at the same time to look through the symbols and transformations to what they stood for (see Figure 2). To paraphrase Bertrand Russell, (formal) algebra allows the user to think less and less about more and more.

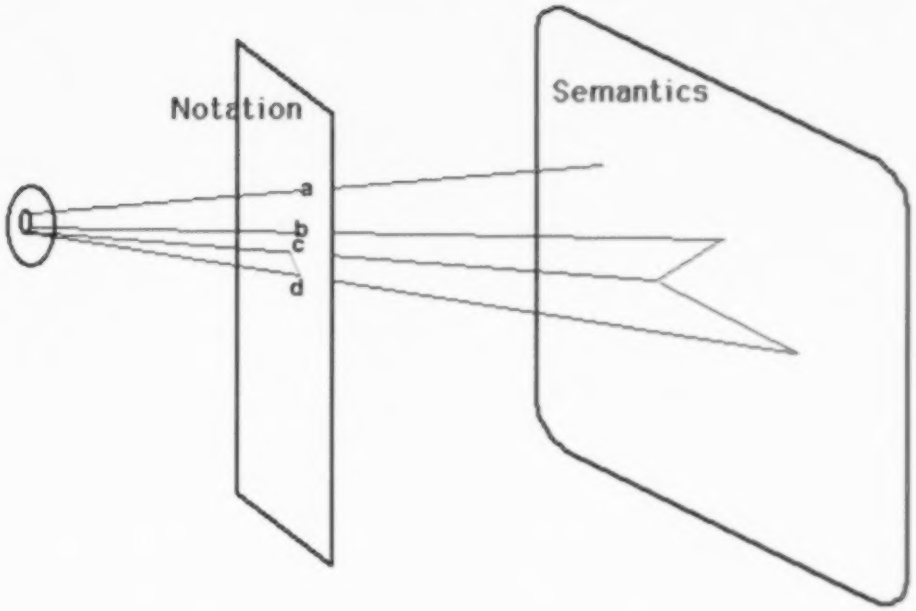


Figure 2. Looking at vs. through symbols.

The problem is that our traditional algebra curriculum has concentrated on the "less and less" part, resulting in many students' inability to see meaning in mathematics and even in their alienation from mathematics. The power of using the form of a mathematical statement as a basis for reasoning is lost when students practice endless rules for symbol manipulation and lose the connection to the quantitative relationships that the symbols might stand for (coming, along the way, to believe that this is what mathematics really is). What happens too often in traditional mathematics classrooms is less learning with understanding than learning with misunderstanding. Research provides many examples of the difficulties into which students have been led when they do not construct their own knowledge or are not given sufficient time to reflect upon what they have learned. (One common problem involves students' overgeneralizing patterns such as linearity, believing, for example, that $(a+b)^2 = a^2 + b^2$ for any a and b. Reflection and trials would convince most students that this pattern does not hold for real numbers except when a or b is zero.) The classroom examples below suggest comparison. The

first illustrates what can (and too often does) happen when students do *not* construct relationships among pieces of mathematical knowledge. The second describes a task taken from a reform curriculum that supports learning with understanding.

Classroom Examples

Common symbol-string misunderstandings. The example discussed here was documented by Guershon Harel (in press). The high school student in this example was attempting to solve the inequality ($(x-1)^2 > 1$). When asked to explain how she arrived at $x > 1$, she responded that "The solution to the equation $(x-1)(x-1) = 0$ is $x = 1$, $x - 1$." She then crossed out the three equality signs and above each wrote an inequality sign >, noting that "x is greater than 1." When she was then asked to solve $(x-1)(x-1) = 3$, she wrote: "$(x-1) = 3$, $(x-1) = 3$." Harel notes that

> [her] mathematical behavior suggests that she was not thinking about the situations [or quantities] that these strings of symbols may represent; rather, the strings themselves were the situations she was reasoning about. That is, Patti's thinking was in terms of a symbolic, superficial structure shared by the three strings. . . . From her perspective, these strings share the same symbolic structure and, therefore, the same solution method must be applicable to them all. (Harel, in press)

Although this example concerns a high school student (mis)solving an inequality ($[x-1]^2 > 1$) because she assumes that equality and inequality behave essentially in the same way, the application of similar procedures to symbols that look alike is common. (Another such example involves "cross-multiplying," a procedure often used blindly without regard to whether the two fractions involved are separated by an equal sign or a plus sign.) For students who reason this way, who appear to be in the majority, not only is the surface shape of a symbol string a call to perform a certain procedure, but dealing with symbol strings (without attaching meaning) is what mathematics is all about. For them, "understanding" is remembering which rules to apply to

which strings of symbols. Unfortunately, understanding algebra requires being able to connect knowledge of procedures with knowledge of concepts.

Meaningful operations on opaque symbols. The example discussed below is taken from a fifth-grade unit, "Patterns & Symbols" (Roodhardt, Kindt, Burrill, & Spence, in press), in Mathematics in Context (National Center for Research in Mathematics and Science & Freudenthal Institute, in press), a National Science Foundation–funded reform curriculum. Among the activities it contains is a task involving transformations on sequences of the letters "S" and "L," where the letters represent rectangular blocks standing on end (S) or lying on their sides (L). In this task (see Figure 3 for example), students work with various transformation rules (e.g., SS→L and LL→S) to act upon such arrays, interpreting their results in terms of strings and vice-versa (e.g., What happens if you repeatedly apply these rules to the above array of blocks?) The students make up their own rules, apply them to their own designs and to those of others, and then interpret them (in both realms). Students gradually move toward more abstract substitution rules, which they can apply to arbitrary strings of symbols (e.g., sequences of their own initials).

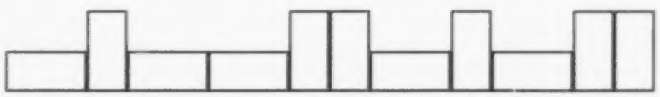


Figure 3. Block array represented by letter sequence (LSLLSSLSLSS).

Summary. Work on (opaque) formalisms is necessary throughout mathematics, independent of topic or students' use of modeling. Tasks such as the one described above both encourage students to work comfortably within a world of opaque symbols not at all based on or referring to numbers and allow students to experience mathematics in ways that encourage understanding rather then alienation.

(Topic-Strand) Algebra as the Study of Structures Abstracted From Computations and Relations

Acts of generalization and abstraction based in computations (where the structure of the computation rather than its result becomes the focus of attention) give rise to abstract structures traditionally associated with "abstract algebra," which, in turn, is traditionally regarded as "fancy" university-level mathematics. This side of algebra, beginning with computations on familiar numbers, has some roots in the 19th-century British idea of algebra as universalized arithmetic, but has deeper roots in number theory. Indeed, this aspect of algebra is precisely what many professional mathematicians mean when they refer to "algebra."

In structural-abstract algebra taught for understanding, structures arise from students' mathematical experience: from matrix representations of motions of the plane, symmetries of geometric figures (see below), modular arithmetic, manipulations of letters in words, or other, fairly arbitrary, even playful contexts. Such structures (a) can be articulated in preformal, natural language, (b) enrich student's understanding of the systems from which they are abstracted, (c) provide students intrinsically useful structures for computations freed of the particulars those structures were once tied to, and (d) provide them a base for yet higher levels of abstraction and formalization. What follows are two classroom examples, one illustrating the use of "natural" language to articulate the structures students discover, and the second, a class inquiry into dihedral group structures.

Classroom Examples

The use of natural language to articulate algebraic structure. In their study of students working a task again from the fifth-grade unit "Patterns & Symbols" (Roodhardt, Kindt, Burrill, & Spence, in press) in Mathematics in Context (National Center for Research in Mathematics and Science & Freudenthal Institute, in press), Spence and Pligge (in press) cited the powerful understanding exhibited by students and their articulation of that understanding in preformal, natural language.

The students had just completed an activity on the concept of even and odd. During this activity, they played a game of "once, twice, go": Two players, on a signal, display a certain number of fingers from one hand. One player wins if the sum of the fingers is even; the other, if the sum is odd. In recording their games, students also used arrays of dots to represent odd and even numbers as well as sums of those numbers. At the end of this activity, they were asked to explain the patterns they could see in these sums. One student explained:

> —An even number and an odd number is always odd. Even always has pairs. Odd always has an extra. Putting them together will still leave that extra, so it's always odd.
>
> —An odd number and an odd number is always even. Odds always have 1 left over, so 2 left over form a new pair.

This student's highly articulate response indicates the power of natural language (even in a fifth grader) to express and justify general relationships, in this case that "an even plus an odd is always odd" and "an odd plus an odd is always even."

Exploration of dihedral groups. In their study of children's natural ability to construct algebraic reasoning, Strom and Lehrer (in press) described a second-grade class using a quilting activity based in the Education Development Center–IBM curriculum unit, Geometry Through Design. This task engages students in a series of ideas customarily associated with the courses in abstract algebra offered to university mathematics majors. The activity begins with students designing a "core square," which is then flipped or rotated to produce four versions of itself in a 2×2 array, the foundation design to be repeated to produce a quilt. Although the students cannot be said to be "doing group theory," they are working in what we could regard as the concrete group of rigid motions of the square. Students working on this task confront many of the issues that university students confront initially when dealing with dihedral groups, such as, What is the operation? When do I know two elements are the same? What is the result of repeatedly multiplying an element by itself? Will I ever get the identity element—the same as not doing anything at all? (This last question leads to the standard group-theory question, "What is the order of the elements of the group?")

Prior to the episode described below, the students had dealt with the issue of when two elements are the same. The class had determined that an "up-flip" (bringing the bottom of the quilting square forward and up) led to the same result as a "down-flip" (bringing the top forward and down) and had decided to call these two actions by the same name, "up-down flip." They then tried to determine how many up-down flips were needed to return a core square to its original position (identified as having the small "x" in the upper-left corner of the side facing them). With teacher scaffolding, they determined that it took two up-down flips to return the core square to its original position. The class then explored what happened when this flip was repeated. Notice that although the teacher (CC) scaffolded the discussion in the example below (and in the ones that follow), it was the students who actually drove the exploration forward, with their own extensions of the ideas and their own conjectures:

CC: And there's her little "x," to mark the top of the core [square], so I know this isn't the flip side. So two up-down flips gets it back right to where it started from.

Na: And zero, um, zero flips.

CC: Zero flips. Yeah, not flipping it.

Br: And four!

CC: Four? Let's try that.

Br Four, six, eight, ten!

CC: Why would two, four, six, eight and ten flips make—

Br: Because, um, because, like, one's an odd number, and two's an even number. So if you just flipped it once it would be—

St: Different!

Br: It'd be the back. So try it four times.

CC: OK. This is Ka's beginning position, the "x" is in the top left. I'm gonna do up-down flips, four of them. Watch what one up-down flip makes it look like [flips the square.] Does it look the same or different?

Br: Different.

CC: OK, now instead of just doing one flip, I'm gonna try four flips. Do you think it will look the same or different?

All: Same.

CC: [Flips the square four times] One. Two. Three. Four.

BR: Brrr-di-doo di-doo! The same!

All: Same, same! [More trumpet sounds and clapping.]

(Strom & Lehrer, in press)

At this point, one student, Br, further conjectured that flipping the square any even number of times would "make the square look the same" as when they started:

CC: Um, what do you think about this idea of Br's? Br's idea is that I could do <u>any even</u> number of flips on this core square—

Br: Can't do eleven, but you can do twelve—

CC: Meaning two, four, six, eight, ten, twelve— Any <u>even</u> number of flips, and it would look the same.

Br: One's odd, two's even, three's odd, four's even.

CC: [Repeating Br to the class] And then she said for you, "One's odd, two's even, three's odd, four's even." OK, that's her idea. Ke has a question for you, Br, about your idea.

Ke: Well, you can go besides by ones, by twos. But if you go by ones, it'll just, like, the square will be on the other side. By twos, you could go up, like, as far as you wanted, and it would still be the same as when it was started, if you go by twos—if you flip it two times.

CC: So if I flip it two times, what will happen, Ke?

Ke: It will be the same.

CC: OK, so you're saying I could do what Br was saying, count by twos, as high as I wanted—

Ke: Then [what I mean is that at any counted number] it would be the same as it is now.

CC: So, no matter how big that number got, if I just counted by twos and then stopped at that number, and then I flipped it that many times, it would look just like this?

All: Yeah. Yes!

(Strom & Lehrer, in press)

Summary. The second graders in the last examples not only dealt with concrete forms of issues and concepts important in elementary group theory, but, in episodes not shown here (see Strom & Lehrer, 1997), they also dealt head-on with questions of argumentation, moving between the particular and the general, and, specifically, the eternal problem of induction from examples. Clearly, the highly skilled orchestration over a long period of time by their teacher is critical to this class's culture of careful inquiry and open discussion. Her pedagogical style created a classroom where students learned mathematical concepts with understanding. The individual quilts created by the students in later extensions of this task not only focused the mathematical thinking, encouraged reflection on what they saw, and provided a common set of tangible, discussible objects, but also made tangible each student's ownership of the problem and the knowledge they themselves constructed. Even more, by design, their quilts shared an underlying structure that was gradually defined and elaborated through their guided discussions. As with the previous examples, this kind of algebra foundation-building is within the reach of most students and teachers, given a classroom culture of teaching and learning for understanding—despite the fact that the mathematics that they are building toward is currently regarded as appropriate for university-level mathematics majors. Finally, the reader may notice that part of this classroom's work appears in Lehrer et al. (this volume), a fact that underscores our point that learning algebraic reasoning is intimately connected to learning other important mathematics.

(Topic-Strand) Algebra as the Study of Functions, Relations, and Joint Variation

The idea of function has perhaps its deepest conceptual roots in our sense of causality, growth, and continuous joint variation—where one quantity changes in conjunction with change in another. Although for most of the 20th century, reform of high school mathematics has called for using function as a central, organizing concept, functions have traditionally been introduced in U.S. schools in precalculus courses at the high school level, and the traditional notation for representing functions has been symbolic (algebraic formulas). Recently, however, headway has been made, primarily in the new reform curricula being developed for middle and high schools.

As the following example illustrates, the concept of function can be fruitfully approached in the early grades, using familiar quantities that change over time (e.g., heights of plants or people, temperature, numbers of people who are eating or asleep at various times throughout the day) and representing them both pictorially and with time-based graphs. Similarly, students can work in familiar contexts. To explore, for example, the cost of beans as a function of the number of packages of beans purchased, students can package the beans themselves, and with appropriate teacher scaffolding, develop along the way their own methods of describing the cost of different numbers of bags of beans.

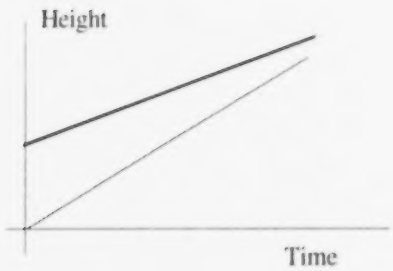
The two ideas of correspondence and variation of quantities, which underlie the concept of function, cut across and unify many different kinds of common mathematical experiences that can readily be introduced in elementary-school classes, including those involved with counting, measuring, and estimating. The idea of function embodies multiple instances, all collected within a single entity (e.g., a list, table, graph), a process which also involves generalizing—answering the question, "What is it that all these instances have in common?"

In the example discussed below (Tierney & Monk, in press), fourth-grade students analyzed graphs of plant height over time, graphs which represented functions of time. Several big ideas associated with interpreting functions came alive in this classroom—without numerical values and without formulas.

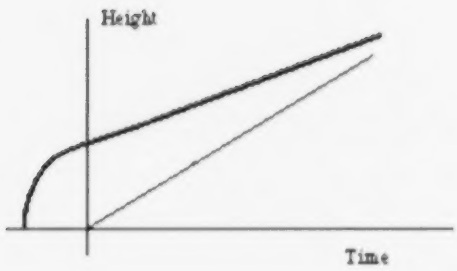
A Classroom Example of Functional Thinking

The example described here was taken from Tierney and Monk (in press). Students, working on a task in the unit "Changes Over Time" (Tierney, Weinberg & Nemirovsky, 1994) from the curriculum series Investigations in Numbers, Data, and Space, were studying height vs. time functions to compare both the changes in the plants' heights and the rate of change of height. Through a symbol system, in this case a graph, and in their own language, students were able to communicate what they learned as they thought, reflected, and talked about vertical height of familiar objects. In so doing, they began to make important distinctions between two ways of looking at functions: the value of a function vs. the rate of change of that value (i.e., height vs. growth rate).

The children grew plants from seed, recorded the growth, and graphed plants' heights each day for two weeks. In the exchanges we look at below, they were interpreting qualitative graphs (provided by the instructor) of plant height over time. In these, only the shapes of graphs were known; no quantities were shown on the axes. In this particular class, all the students interpreted the steeper graphs as meaning the plant was growing faster and the higher graphs as showing a taller plant. At the point of the discussion below, students were working on a problem with two graphs: the first above the second, but not as steep; the second lower but steeper (see Figure 4a). This "crossing-difference" between the respective height and steepness properties of the two graphs provoked disagreement based in distinguishing height from change of height and change from rate of change. Some students focused on change in height (depicted by the slope of the graph) whereas others focused on current height and the growth of the plant that yielded the beginning height (before the growth shown on the graph [see Figure 4b]). What follows is a description of the exchanges as students worked to understand what they saw and resolve it with what they already believed.



(a)



(b)

Figure 4. Using height vs. time functions to compare both changes in the plants' heights and the rate of change of height. Note that (b) shows the time "before the beginning" of (a).

When the teacher asked which plant was growing faster, one student compared the growth of the plants by comparing the changes in height in a fixed amount of time (the rates of growth): "The light line. It started really small and got bigger and bigger and took the same amount of time to get to the same height." Two other students, however, responded directly to the shape, interpreting it in terms of comparative change:

> "The light one [grows faster], because it's always going up. The dark one is kind of steady and kind of going across."
>
> "The dark one is slightly going up and its not going fast."

When questioned by a peer, this last student changed to another student's approach: "It didn't grow high in a short time." When the teacher questioned him ("Tell me about the changes"), he based his answer on the shape of the line, describing it in a language appropriate for the plant it depicted: "The dark line is only growing a little bit over a long time. The light line, the changes are bigger over the same amount of time."

Two other students then talked about the plant before and after the time depicted on the graph (see Figure 4b):

> "I chose the dark line. The light line takes time to grow up. It's going to take it a long time to catch up with the black line."
>
> "The dark line grew faster at the beginning, before the graph."

The teacher asked the second of these students to come up to the board and draw the dark line as he thought it might have been before the graph began. He started at the left end of the dark line and extended it leftward, making a line that curved down to the horizontal axis, almost vertically. The student, moving his finger along the line he drew, said that "It grew fast, then still fast, then started to get steady."

Summary. The excerpts above were part of what Tierney and Monk (in press) described as a spirited discussion among students exploring the concept of function: in this case, the properties of the functions describing the height over time of plants they themselves had grown and how those properties might relate to what the functions stood for. The many ways of comparing change and interpreting graphs (explored as well in university calculus courses) appeared in the informal language and graphical notation of these fourth graders. Their understanding of how plants grow, refined in the two weeks of recording the growth of their own plants (previous to this discussion), was extended and applied to make sense of the graphs and relations among sizes of plants, changes in size, and rates of change. To decide which plant was growing faster, some children focused on visual aspects, such as steepness, whereas others focused on implicit quantitative information.

One student's belief that "the dark line grew faster at the beginning, before the graph" (i.e., that part of the graph was missing) raises an important issue in the use of symbols, in this case, the representation or model: "Is it a complete record, the only source of available information about the event, or is it like an illustration that tells part of the story to be supplemented by other things we know and believe?" (Tierney & Monk, in press).

How do the ways we organize the representation system (into parts and wholes) relate to the way we organize the experience being represented? Does the whole graph always stand for a whole situation, or, if not, when does it stand for only part of one? (And how can we tell when it does?) Does the labeling of the graph give it absolute meaning/range? Or can we assume, as this student did, that there is a before-the-beginning and an after-the-end? These are deep, subtle matters—and, importantly, elementary school students, in an extended exploration in an investigative classroom culture (i.e., a classroom where students learn with understanding) can begin to make sense of such mathematical issues.

Algebra as a Cluster of Modeling and Phenomena-Controlling Languages

Quantitative reasoning, as well as the use of functions and relations, involves building mathematical systems through, usually, several cycles of improvement and interpretation, which act to describe phenomena or situations and to support reasoning about them. Put more simply, quantitative reasoning involves modeling, and many have argued that the modeling of situations is the primary reason for studying algebra.

In modeling, we begin with phenomena and attempt to mathematize them. But the use of computers and graphing calculators, increasingly common in nontraditional and some traditional classrooms, enables us to rethink how we explore and model phenomena—and how we can assist students in coming to understand the mathematical concepts behind those phenomena. For example, we can now use mathematics to simulate phenomena within the computer and even drive physical devices such as motorized cars on a track using data from a computer or graphing calculator. In fact, computer languages amount to an algebra-like language within which we can create, explore (to some degree "experience"), and extend mathematical environments. Similarly,

coordinate graphs can create/control phenomena. Whether these technological environments are used to model or create/control phenomena, they change in fundamental ways how we relate the particular to the general and how we can state and justify mathematical conjectures. But more importantly, they can change how we teach and learn mathematics— even, in fact, how we relate to the mathematics itself. The example below suggests a new level of intimacy between students' activity and the mathematical notations that they use and interpret.

Example of Mixed Modeling and Phenomenon-Controlling Interactions in Physical and Computer-Based Motion

Background. As part of a 5-week summer program for economically disadvantaged children, 15 students, who had recently finished either third or fourth grade, were involved in a program conducted by two teachers and the principal from the students' (urban) school and who were assisted by the author and two SimCalc[2] project staff members. Students were involved in an extended exploration first of their own physical movement and then of related movement issues in a computer simulation (developed as part of the SimCalc project [directed by the author]). Students' work began with creating a 50-ft path with masking tape in the gymnasium and then marking it at 2-ft intervals, with double marks at the tens' places. Over a period of three days, they studied their own motion, using a combination of the marked masking-tape path and stopwatches. Although (as expected) they were unable to quantify their velocity numerically, they were quite able to distinguish three values of their own speed: "slow," "medium," and "fast." They also accepted the fact that one person's "medium" might be close to someone else's "slow" or "fast." They timed one another's "trips" down the "path" and recorded these in three tables, one each for "slow," "medium," and "fast." The fact that they were able to move, measure, count, and record their data was a source of delight and fascination for them.

Students then moved to a motion-simulation software system, "Elevators," part of the SimCalc MathWorlds software system (Kaput, Roschelle, DeLaura, Burke, & Zeppenfeld, 1997). Through "elevators" that they themselves were able to control, using velocity vs. time graphs, students were able to reenact a number of the activities that they had engaged in

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physically. The sections that follow describe how they related their own physical and kinesthetic experiences to their computer-based ones (elevator simulations) over eight sessions spread over two hot (and un-air-conditioned) summer weeks.

Building understanding. On the third day (after they had recorded the lengths of time they took to move the entire 50-foot pathway at their "slow," "medium," and "fast" paces and had discussed how these rates differed from student to student), students engaged in planned movements: to walk or run for given lengths of time (timed with a stopwatch) at various paces. For example, a student might be asked to go for two seconds each at "slow," "fast," and "medium," or to go fast for two seconds, slow for three, and medium for two more. Several students repeated a given set of directions, stopping and standing at the end of their "trip" as a way of recording and comparing (with their peers) how far they had gone under the given stipulations. The boys, more than the girls, tended to value higher speeds, hence distances, particularly early in the activities. The issue of competition arose as some students came to realize that the purpose of the activity was not to go as far as possible under the given velocity constraints, but to be as precise as possible in carrying out the motion instructions. These activities helped the students develop a perspective on their own motion and a sensitivity to relationships among time, speed, and distance— although they were not expected to quantify these relationships until much later, after substantial experience.

On the third and fourth days, students engaged in a paired activity, in which one student of the pair was to move at a constant rate while the other moved according to directions similar to ones that they had previously enacted. After a couple of trials, during which the students started and ended at the same times, the "constant speed" student was asked to travel at a perfectly constant rate in such a way as to end up at exactly the same place as the other student by the end of the given time interval. Thus, if successful, the constant-speed student would be moving at the average speed of his/her counterpart, whose speed would vary according to the given instructions (some of which were provided by other students).

Bringing out the notion of constant speed in this very concrete, visible way directly confronted students' tendencies to want to "catch up." Those students who were not in the currently moving pair lined the pathway, observing the motion closely. When a student, who was supposed to be traveling at a constant speed, speeded up or slowed down (usually in fear that he or she would not reach the end point simultaneously with his or her counterpart), the "audience" shouted its disapproval or corrections. This led to considerable concentration on the part of those moving to maintain a constant pace.

For students who cycled through this activity and variations of it (eventually all students), the notion of constant rate became constructed knowledge, a notion of speed they were acutely sensitive to and were able to reflect on and articulate. Students discussed and were concerned, for example, that the person moving reach the given constant speed as soon as possible, and that, at the end of the "trip," the moving stop abruptly. Because other students were timing them as they carried out the publicly announced slow–medium–fast instructions, they also became more aware of the length of a given time interval: The quantities and concepts associated with constant speed and time interval were made their own in an intimate, deeply felt kinesthetic sense and , at the same time, were connected to their emerging knowledge of graphs and functions.

Using computer simulations and dynamic graphs to represent quantitative relationships. At first, the quantitative relationships and concepts arising from these activities were described using preformal language and numbers. Beginning on the fifth day, however, the students moved to the computer simulations: buildings with two elevator shafts each. The screen had three draggable icons on the toolbar: one each for slow, medium, and fast (see Figure 5). These students could drop on a velocity vs. time graph and get a horizontal line segment that represented either one, two, or four floors per second in the simulation. They could then stretch these segments horizontally as needed to determine the length of time the elevator traveled at the indicated speed. (Thus, students could create and manipulate "piecewise constant velocity" functions.) When the students "ran" the elevators (by clicking on the big arrow in the Controls window), the elevators moved according to the velocity graphs that the students had created.

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Working in pairs at the computer over the fifth and sixth days, the students worked through a series of activities that paralleled and then built upon the work they had done physically in the gymnasium.

The point of the computer side of the activity, however, was to continue the process of quantifying, or mathematizing, motion. Physical and cybernetic environments differ in a fundamental way: The physical is kinesthetically rich, but quantitatively poor (or, perhaps more accurately, inaccessible); the cybernetic is kinesthetically vacuous, but quantitatively rich. For example, the floors in the building are numbered, and the graphs are numbered and labeled in such a way as to help students establish a close connection between the graphs and the building floors. Thus, a student creating a "medium" velocity graph (2 floors per second), with the elevator traveling a given direction for a given length of time (6 seconds—as in the flat graph in Figure 5) would be asked, "Where will the elevator finish its trip?" Rather quickly, the students came to see that the area (a rectangle) under the flat-topped graph segment gives the answer, and that, if the velocity is composed of several segments, they could predict the final position by adding up the areas under the appropriate segments.

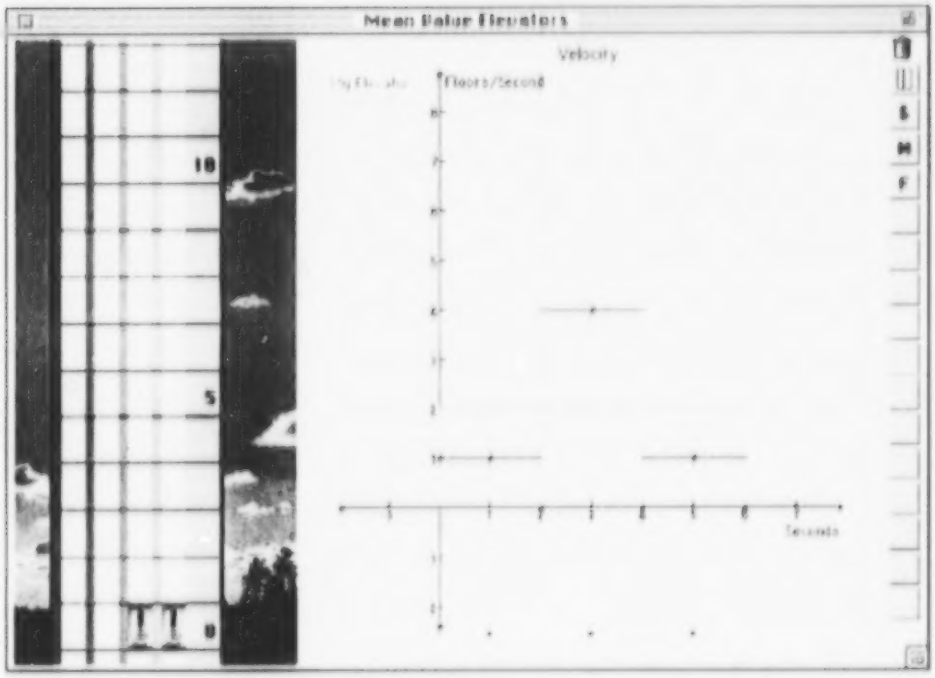

Figure 5. Slow, medium, fast velocity graphs controlling the elevators in SimCalc task.

In the final two days, the students returned to the physical context to deal with the issues of changing direction and negative velocity—repeating the cycle of physical and then computer-based motion. Somewhat surprising was the ease with which students made the transition between horizontal and vertical motion. (At that time, we did not have a horizontal motion simulation available.) It appears that the natural motion-talk, which occurred in both contexts, served to link the two motions. Discussions about "speeding up," "slowing down," "turning around," and so on applied to both realms, and these terms were used to describe both the graphs and the motion (both the motion on the computer screen and their own physical motion).

Summary. Students were quite clearly personally invested in the simulation and the motions they were creating. Almost unanimously they chose to name the elevators after themselves (the program allowed them to both color and name the elevators, and new versions of the software allow students to modify actors in the simulations even further). When pairs of

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motions were involved, such as in the average-speed activity, they took ownership of their respective elevators, often referring to them in the first person (as in "I'm ahead" or "I'm slowing down"). Additionally, the fact that they worked on the computer in pairs allowed them to continue to communicate informally— an extension of the conversational mode employed in the physical context.

The development of understanding in this situation involved intimate connections between students' physical actions and the motion simulations, connections that were mediated by the students' own talk about both sides of the connection and by the not-quite-standard graphical notation, which proved a powerful form of expression for the students, both as a modeling language and as a phenomenon-controlling language. We feel that the structure of the activity and students' own (similar) physical exploration of motion were the foundations both for their understanding and their investment—their making this knowledge their own. Without that prior physical exploration, which built the concepts and kinesthetic sense of what these velocity graphs were all about, this would have been an empty exercise in symbol manipulation.

Reflections

The five aspects of algebra except for the second (symbol manipulation) that we have examined in this chapter are not well represented in standard algebra courses. Some may be tempted to say that what we have described here is not algebra, to which we would reply, "Yes," if "algebra" is what commonly occurs in standard Algebra I and Algebra II courses. But the assumption of this chapter is that algebra must be much broader, deeper, and richer than that. Algebra writ large cuts across topics and adds a conceptual unity, depth, and power that in our K–8 curriculum, especially in the earlier grades, has been difficult to achieve. Although in this chapter we have deliberately chosen illustrations that exhibited young students led by sensitive teachers, the algebra we have shown here is neither a mystery nor out of reach of most teachers and most students. Indeed, elementary and middle-school students can make sense of complex situations while simultaneously building big mathematical ideas. It is our premise that this

"fancy" algebra, taught, we hope, in classrooms that promote understanding, will prove more accessible than the traditional algebra that everybody loves to hate.

Footnotes

1. Funded by the NSF Applications of Advanced Technology Program, Grant #RED 9619102 and the National Center for Improvement of Student Learning and Achievement, University of Wisconsin–Madison, U.S. Dept. of Education Prime Grant #R305A60007.

2. SimCalc is a technology and curriculum research and development project intended to democratize access to the basic ideas underlying calculus beginning in the early grades and extending to AP calculus and beyond.

For Further Reading

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